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# Chern maps for $U$-theory

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**Abstract**

For an exact $k$-linear category $\mathscr{A}$ with a duality functor, the dihedral homology of $\mathscr{A}$ is defined. We show that when $\frac{1}{2} \in k$, the image of the generalized Chern map from the $U$-theory of $\mathscr{A}$ to cyclic homology, lies in a direct summand which is the dihedral homology of $\mathscr{A}$. When $\mathscr{A}$ is the category of finitely generated projective $A$-modules over a $k$-algebra $A$, we recover the early results of Cortinas (1993) and Lodder (1992). The approach we adopt here is more categorical, inspired by McCarthy's work (1992).

## 0. Introduction

All rings (algebras) are assumed to be unitary and all modules right modules if not otherwise specified. All categories are assumed to be small categories.

Let $A$ be a $k$-algebra where $k$ is a commutative ring. The cyclic homology and cohomology of $A$ was first developed by A. Connes as a non-commutative substitute of De Rahm cohomology and he constructed a Chern character map with values in it. Jones and Goodwillie generalized this to a Chern map which has its values in negative homology.

Suppose $A$ has an involution $a \to \bar{a}$ which acts trivially on $k$, i.e., $\bar{\bar{a}} = a, \overline{a+b} = \bar{a} + \bar{b}, \overline{ab} = \bar{b}\bar{a}$ and $\overline{a\alpha} = \bar{a}\alpha \ \forall \alpha \in k$. The involution can be used to construct another variant of cyclic homology, which is called dihedral homology. It is shown in [2, 9] that if $\frac{1}{2} \in k$, then the composition, forgetful map from $L$-theory to $K$-theory followed by the generalized Chern map, has its image in dihedral homology.

A categorical approach was instituted for cyclic homology by McCarthy (see [10]) and as in the case of $K$-theory it helped shed new light on the subject. In particular he constructed the generalized Chern map, the so-called Jones–Goodwillie Chern map, in a straightforward manner.

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In this paper we are going to show that we can do the same for a $k$-linear exact category.

If $\mathscr{A}$ is a $k$-linear category with a duality functor, we can construct dihedral homolgy for it. If the category happens to be the finitely generated projective modules over a $k$-algebra with involution, then we show that the categorical definition agrees with the usual one.

If $\mathscr{A}$ is an exact $k$-linear category with a duality functor such that all short exact sequences split, then copying Quillen's $S^{-1}S$ construction, we can extend the definition of $L$-theory to such a category. In that case we define a generalized Chern map from $L$-theory to cyclic homology. In Theorem 4.5 we show that if $\frac{1}{2} \in k$, then the generalized Chern map lies in dihedral homology. In particular this gives a new categorical proof of the result in [2, 9] mentioned above. For the general case of an exact category we will look at another part of Karoubi's $L$-theory.

For a $k$-algebra $A$ with involution, the forgetful map and hyperbolic map between the $K$-theory and $L$-theory provide basic relations between these two theories. M. Karoubi is the first to systematically study these two maps and he calls the homotopy fiber of the hyperbolic map $K(A) \to {}_{\varepsilon}L(A)$ the $U$-theory of $A$, denoted as ${}_{\varepsilon}U(A)$. This fibration has been used to derive many results for these theories. This $U$-theory also has a categorical construction [1, 12, 14]. Unlike the $S^{-1}S$-constructions of $L$-theory, which only makes good sense when the short exact sequences in the exact category are all split, the $U$-theory generalizes well to any exact category with a duality functor. From this point of view, $U$-theory seems more natural than $L$-theory.

For an exact $k$-linear category with a duality functor we define the generalized Chern map from $U$-theory to cyclic homology and show (Theorem 4.3 and Corolary 4.4) that if $\frac{1}{2} \in k$, the image of the map lies in the dihedral homology.

Here is a brief account of the contents. In Section 1 we use Quillen's $Q$-construction to define cyclic homology and its variants for a $k$-linear category. This is comparable to the construction in [10] which uses the Waldhausen $S$-construction. In Section 2 we review the definitions of $U$-theory for an exact category with a duality functor and in Section 3 we show how to define dihedral homology for such categories. In Section 4 the two main results of this paper, Theorems 4.3 and 4.5, are proved.

## 1. $Q$-Construction for cyclic homology and its variants

Let $A$ be a $k$-algebra where $k$ is commutative ring, and let $\bar{A} = A/k$. As usual $A^{\otimes n}$ (resp. $\bar{A}^{\otimes n}$) will denote the tensor product of $n$ copies of $A$ (resp. $\bar{A}$) over $k$.

Let $C_n(A) = A \otimes A^{\otimes n}$ and let $d_i : C_n(A) \to C_{n-1}(A)$ be

$$d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \quad 0 \le i \le n-1,$$

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_n a_0 \otimes \cdots \otimes a_{n-1},$$

and for $0 \le i \le n$ let $s_i : C_n(A) \to C_{n+1}(A)$ be

$$s_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

Then $(C(A), d_i, s_i)$ is a simplicial $k$-module. There is another degeneracy map (usually called the extra degeneracy) $s : C_n(A) \to C_{n+1}(A)$ which is given by $s(a_0 \otimes \cdots \otimes a_n) = 1 \otimes a_0 \otimes \cdots \otimes a_n$.

The Hochschild homology of $A$, $HH_*(A)$, is defined to be homology of the chain complex $(C(A), b)$ where $b = \sum_{i=0}^{n} (-1)^n d_i$.

Let $\bar{C}_n(A) = A \otimes \bar{A}^{\otimes n} = C_n(A)/D_n$, where $D_n(A)$ is the sub $k$-module of $C_n(A)$ generated by all elements of the form $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ where $a_i = 1$ for some $1 \le i \le n$. Equivalently, $D_n$ is the sub $k$-module generated by the images of the degeneracy maps $s_i$. The chain complex $(\bar{C}(A), b)$ is called the normalization of $(C(A), b)$ and it is well known that they have the same homology. For a ring with involution (resp. a category with a duality functor), it is $(\bar{C}(A), b)$, not $(C(A), b)$, that behaves well with respect to the involution (resp. duality functor). So we will mainly work with normalization and think of $HH_*(A)$ as $H_*(\bar{C}(A), b)$.

To define the cyclic homolgy of $A$, let $t : C_n(A) \to C_n(A)$ act by

$$t(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}$$

and let $N = 1 + t + \cdots + t^n$. In the normalization $\bar{C}(A)$, the Connes boundary map (which we still denote as $B$) is given by $B = s \circ N$, and satisfies $B \circ b + b \circ B = 0$ (see [7]). This yields the following double chain complex:

$$\begin{array}{ccccccc}
 & \downarrow b & & \downarrow b & & \downarrow b & \\
\xleftarrow{B} & \bar{C}_3(A) & \xleftarrow{B} & \bar{C}_2(A) & \xleftarrow{B} & \bar{C}_1(A) & \xleftarrow{B} \\
 & \downarrow b & & \downarrow b & & \downarrow b & \\
\xleftarrow{B} & \bar{C}_2(A) & \xleftarrow{B} & \bar{C}_1(A) & \xleftarrow{B} & \bar{C}_0(A) & \\
 & \downarrow b & & \downarrow b & & & \\
\xleftarrow{B} & \bar{C}_1(A) & \xleftarrow{B} & \bar{C}_0(A) & & & \\
 & \downarrow b & & & & & \\
\xleftarrow{B} & \bar{C}_0(A) & & & & & \\
 & p = -1 & & p = 0 & & p = 1 &
\end{array}$$

We use $B\bar{P}(A)$ to denote the total complex of the above double complex, $B\bar{C}(A)$ to denote the total complex of the double complex which contains those columns where $p \ge 0$, and $B\bar{N}(A)$ to denote the total complex of the double complex which contains those columns where $p \le 0$. From these three chain complexes we get cyclic homology and its variants:

$HP_*(A) = H_*(B\bar{P}(A))$ periodic homology,

$HC_*(A) = H_*(B\bar{C}(A))$ cyclic homology,

$HN_*(A) = H_*(B\bar{N}(A))$ negative homology,

For more details see [7].

A similar procedure combined with Waldhausen's $S.$-construction was used by McCarthy in [10] to define these homologies categorically as follows.

Suppose $\mathscr{A}$ is a $k$-linear category, i.e., for all $A, B \in \mathscr{A}$, $Hom(A, B)$ is a $k$-module, and compositions of morphisms are $k$-bilinear. Let

$$C_n(\mathscr{A}) = \bigoplus Hom_{\mathscr{A}}(A_1, A_0) \otimes_k Hom_{\mathscr{A}}(A_2, A_1) \otimes_k \cdots \otimes_k Hom_{\mathscr{A}}(A_0, A_n)$$

where the direct sum runs over all $(A_0, A_1, \ldots, A_n) \in Obj(\mathscr{A})^{n+1}$. We have maps $d_i$, $s_i$, given by

$$d_i(f_0 \otimes \cdots \otimes f_n) = \begin{cases} f_0 \otimes \cdots \otimes f_i \circ f_{i+1} \otimes \cdots \otimes f_n & \text{if } 0 \le i \le n-1, \\ f_n \circ f_0 \otimes f_1 \otimes \cdots \otimes f_{n-1} & \text{if } i = n, \end{cases}$$

and

$$s_i(f_0 \otimes \cdots \otimes f_n) = \begin{cases} f_0 \otimes \cdots \otimes f_i \otimes id_{A_{i+1}} \otimes f_{i+1} \otimes \cdots \otimes f_n & \text{if } 0 \le i \le n-1, \\ f_0 \otimes \cdots \otimes f_n \otimes id_{A_0} & \text{if } i = n. \end{cases}$$

Then $(C(\mathscr{A}), d_i, s_i)$ is a simplicial $k$-module and $(C(\mathscr{A}), b)$ is a chain complex, where $b = \sum_{i=0}^{n} d_i$.

We then have the map given by $t(f_0 \otimes \cdots \otimes f_n) = (-1)^n (f_n \otimes f_0 \otimes \cdots \otimes f_{n-1})$. Then $(C(\mathscr{A}), d_i, s_i, t)$ is a cyclic $k$-module. Let $N = \sum_{i=0}^{n} t^i$. Finally we have the extra degeneracy map:

$$s(f_0 \otimes \cdots \otimes f_n) = id_{A_0} \otimes f_0 \otimes \cdots \otimes f_n.$$

As in the ring case let $\bar{C}_n(\mathscr{A}) := C_n(\mathscr{A})/D_n$, where $D_n$ is the sub $k$-module generated by the images of the degeneracy maps $s_i$ (Note: the special degeneracy map $s$ is not included). This gives us the normalized chain complex $(\bar{C}(\mathscr{A}), b)$ (we do not change the notation and still write the map as $b$). Let $B = s \circ N : \bar{C}_n(\mathscr{A}) \to \bar{C}_{n+1}(\mathscr{A})$. Then we have $b \circ B + B \circ b = 0$. So we can form the double complex:

$$\begin{array}{ccccccc}
 & b\downarrow & & b\downarrow & & b\downarrow & \\
\xleftarrow{B} & \bar{C}_3(\mathscr{A}) & \xleftarrow{B} & \bar{C}_2(\mathscr{A}) & \xleftarrow{B} & \bar{C}_1(\mathscr{A}) & \xleftarrow{B} \\
 & b\downarrow & & b\downarrow & & b\downarrow & \\
\xleftarrow{B} & \bar{C}_2(\mathscr{A}) & \xleftarrow{B} & \bar{C}_1(\mathscr{A}) & \xleftarrow{B} & \bar{C}_0(\mathscr{A}) & \\
 & b\downarrow & & b\downarrow & & & \\
\xleftarrow{B} & \bar{C}_1(\mathscr{A}) & \xleftarrow{B} & \bar{C}_0(\mathscr{A}) & & & \\
 & b\downarrow & & & & & \\
\xleftarrow{B} & \bar{C}_0(\mathscr{A}) & & & & & \\
 & p=-1 & & p=0 & & p=1 &
\end{array}$$

As before $B\bar{P}(\mathscr{A})$ will denote the total complex of the above double complex, $B\bar{C}(\mathscr{A})$ will denote the total complex of the double complex which contains those columns where $p \ge 0$, and $B\bar{N}(\mathscr{A})$ will denote the total complex of the double complex which contains those columns where $p \le 0$.

Recall Waldhausen's $S.$-construction for an exact category $\mathscr{A}$ [15]. $S.\mathscr{A}$ is a simplicial category where $S_n(\mathscr{A})$ is the category whose objects are all the sequences of admissable monomorphisms of length $n$, along with choices for all quotients:

$$* \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n \quad \text{(we supress writing the choices for quotients)},$$

and morphisms are the obvious ones. If $\mathscr{A}$ is an exact $k$-linear category, then each $S_n(\mathscr{A})$ is a $k$-linear category. Hence we can apply the above constructions $\bar{C}$, $B\bar{C}$, $B\bar{P}$, $B\bar{N}$ to each $S_n(\mathscr{A})$. Since they are all functorial, we have simplicial complexes, $\bar{C}(S.\mathscr{A})$, $B\bar{C}(S.\mathscr{A})$, $B\bar{P}(S.\mathscr{A})$ and $B\bar{N}(S.\mathscr{A})$. Hence each can be made into a double chain complex. From these double chain complexes we can define:

$$HH_*(\mathscr{A}) = H_{*+1}(Tot(\bar{C}(S.\mathscr{A}))) \text{ Hochschild homology},$$

$$HP_*(\mathscr{A}) = H_{*+1}(Tot(B\bar{P}(S.\mathscr{A}))) \text{ periodic homology},$$

$$HC_*(\mathscr{A}) = H_{*+1}(Tot(B\bar{C}(S.\mathscr{A}))) \text{ cyclic homology},$$

$$HN_*(\mathscr{A}) = H_{*+1}(Tot(B\bar{N}(S.\mathscr{A}))) \text{ negative homology}.$$

It is shown in [10] that when $A$ is a $k$-algebra and $\mathscr{P}_A$ is the exact $k$-linear category of all finitely generated projective $A$-modules, $HH_*(A) = HH_*(\mathscr{P}_A)$ and the same is true for $H\bar{P}_*$, $H\bar{C}_*$ and $H\bar{N}_*$.

The Dennis trace map and the Jones–Goodwillie Chern maps are defined in [10]. Furthermore, it is shown in [10] that when $\mathscr{A}$ is $\mathscr{P}_A$, where $A$ is a $k$-algebra, these two maps are the same as the usual maps for the algebra $A$.

The Dennis trace map $D: K_*(\mathscr{A}) \to HH_*(\mathscr{A})$ is induced by the map

$$S.\mathscr{A} \xrightarrow{id} \bar{C}_0 S.\mathscr{A} \xrightarrow{inc} \bar{C}S.\mathscr{A}$$

where $id$ sends each $A \in S_n(\mathscr{A})$ to $id_A \in Hom_{S_n(\mathscr{A})}(A, A) \subset \bar{C}_0 S_n(\mathscr{A})$.

The Jones–Goodwillie Chern map, denoted as $J - G: K_*(\mathscr{A}) \to HN_*(\mathscr{A})$, is obtained by the composition

$$S.\mathscr{A} \xrightarrow{\alpha} Z_0(B\bar{N}S.\mathscr{A}) \to B\bar{N}S.\mathscr{A},$$

where $\alpha$ sends each $A \in S_n(\mathscr{A})$ to $(id_A, 0, 0, \ldots)$, and $Z_0(B\bar{N}S.\mathscr{A})$ are the 0-cycles of $B\bar{N}S.\mathscr{A}$.

For more details and other wonderful properties see [10].

For our purpose we need a Quillen's $Q$-construction for $HH_*$, $HP_*$, $HC_*$, $HN_*$ of an exact $k$-linear category. Let us first recall Quillen's $Q$-construction.

Let $\mathscr{A}$ be an exact category. $Q\mathscr{A}$ is the category which has the same objects as $\mathscr{A}$, but for defining the morphisms in $Q\mathscr{A}$ let us introduce some equivalences. Two diagrams in $\mathscr{A}$

$$M \twoheadleftarrow M' \rightarrowtail N \quad \text{and} \quad M \twoheadleftarrow M'' \rightarrowtail N$$

are said to be equivalent if there is an isomorphism $\theta : M' \to M''$ such that the diagrams

$$\begin{array}{ccccc} M & \twoheadleftarrow & M' & \rightarrowtail & N \\ \| & & \downarrow{\scriptstyle \theta} & & \| \\ M & \twoheadleftarrow & M'' & \rightarrowtail & N \end{array}$$

commute. Then a morphism in $Q\mathscr{A}$ from $M$ to $N$ is defined to be an equivalence class of a diagram $M \twoheadleftarrow M' \rightarrowtail N$, for some $M'$ in $Q\mathscr{A}$.

In $\mathscr{A}$ two admissible monomorphism $N_1 \rightarrowtail N$ and $N_2 \rightarrowtail N$ are said to be equivalent if there is an isomorphism $\theta : N_1 \to N_2$ such that

$$\begin{array}{ccc} N_1 & \rightarrowtail & N \\ {\scriptstyle \theta}\downarrow & \nearrow & \\ N_2 & & \end{array}$$

commutes. Then a subobject $N_1$ of $N$ is an equivalence class of an admissible monomorphism $N_1 \rightarrowtail N$, denoted by $N_1 \subset N$. Using this we can specify a morphism in $Q\mathscr{A}$ from $M$ to $N$ by $(N_1, N_2, \varphi)$ where $N_1 \subset N_2 \subset N$ and $\varphi : N_2/N_1 \to M$ is an isomorphism.

Consider now the bicategory, $bi(Q\mathscr{A})$, whose objects are the objects of $\mathscr{A}$, horizontal morphisms are morphisms in $Q\mathscr{A}$, vertical morphisms are morphisms in $\mathscr{A}$, and bimorphisms are all diagrams:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ P & \xrightarrow{g} & Q \end{array}$$

where $f$, $g$ are morphisms in $Q\mathscr{A}$, $a$, $b$ are morphisms in $\mathscr{A}$, and the diagram commutes in the following sense.

If $f$ is denoted by $N_1 \subset N_2 \subset N$, $\varphi : N_2/N_1 \xrightarrow{\simeq} M$ and $g$ is denoted by $Q_1 \subset Q_2 \subset Q, \psi : Q_2/Q_1 \xrightarrow{\simeq} P$, then $f | N_2 : N_2 \to Q_2, f | N_1 : N_1 \to Q_1$ and the diagram

$$\begin{array}{ccc} N_2/N_1 & \xrightarrow{\varphi} & M \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle a} \\ Q_2/Q_1 & \xrightarrow{\psi} & P \end{array}$$

commutes.

Let $Q.\mathscr{A}$ be the horizontal nerve of $bi(Q\mathscr{A})$. More precisely, $Q.\mathscr{A}$ is the simplicial category $(Q_n\mathscr{A}, d_i, s_i)$ as follows. $Q_n\mathscr{A}$ is the category whose objects are all sequences of length $n$ in $Q\mathscr{A}$

$$M_0 \to M_1 \to \cdots \to M_n$$

and a morphism between two such sequences is a family of vertical arrows each of which is a morphism in $\mathscr{A}$

$$\begin{array}{ccccccc} M_0 & \longrightarrow & M_1 & \longrightarrow & \cdots & M_n \\ \downarrow & & \downarrow & & & \downarrow \\ P_0 & \longrightarrow & P_1 & \longrightarrow & \cdots & P_n \end{array}$$

such that each square commutes in the above sense. The face maps and the degeneracy maps are the same as those for the nerve of $Q.\mathscr{A}$.

Clearly when $\mathscr{A}$ is a $k$-linear category, each $Q_n\mathscr{A}$ is also a $k$-linear category. Therefore, we can apply the constructions $\bar{C}$, $B\bar{C}$, $B\bar{N}$ and $B\bar{P}$ to each $Q_n\mathscr{A}$. Then we can apply them to $Q.\mathscr{A}$ to obtain double chain complexes. With this we can define homology by

$HH_*(\mathscr{A}) = H_{*+1}(\bar{C}(Q.\mathscr{A}))$ Hochschild homology,

$HP_*(\mathscr{A}) = H_{*+1}(B\bar{P}(Q.\mathscr{A}))$ periodic homology,

$HC_*(\mathscr{A}) = H_{*+1}(B\bar{C}(Q.\mathscr{A}))$ cyclic homology,

$HN_*(\mathscr{A}) = H_{*+1}(B\bar{N}(Q.\mathscr{A}))$ negative homology.

**Proposition 1.1.** *Let $\mathscr{A}$ be an exact k-linear category. Then the two definitions, one by the Q.-construction as above, and the other by the S.-construction as given by McCarthy, of the Hochschild, periodic, cyclic and negative homologies of $\mathscr{A}$ are the same. In particular, if A is a k-algebra then*

$$HH_*(A) = H_{*+1}(\bar{C}(Q.\mathscr{P}_A)), \qquad HP_*(A) = H_{*+1}(B\bar{P}(Q.\mathscr{P}_A)),$$

$$HC_*(A) = H_{*+1}(B\bar{C}(Q.\mathscr{P}_A)), \qquad HN_*(A) = H_{*+1}(B\bar{N}(Q.\mathscr{P}_A))$$

*where $\mathscr{P}_A$ is the category of all finitely generated projective A-modules.*

**Proof.** We mimick Waldhausen's proof that the two definitions of $K$-theory of an exact category by the $Q$-construction and the $S$.-construction are equivalent [15, 1.9]. For the simplicial chain complex $\bar{C}(S.\mathscr{A})$ let $\bar{C}(S^e.\mathscr{A})$ denote the corresponding edgewise subdivision. As in [15] it is not too hard to check that for each $n$, the $k$-linear category $S^e_n\mathscr{A} = S_{2n+1}\mathscr{A}$ is equivalent to the $k$-linear category $Q_n\mathscr{A}$. So there is a special homotopy equivalence between the chain complexes $\bar{C}(S^e_n\mathscr{A})$ and $\bar{C}(Q_n\mathscr{A})$ [10, proposition 2.4.1]. Therefore we have a homotopy equivalence between the total complexes of $\bar{C}(S^e.\mathscr{A})$ and $\bar{C}(Q.\mathscr{A})$. From this we have

$$H_*(\bar{C}(Q.\mathscr{A}) \cong H_*(\bar{C}(S^e.\mathscr{A})) \cong H_*(\bar{C}(S.\mathscr{A})).$$

There are similar proofs for $HP_*$, $\mathrm{HC}_*$ and $HN_*$. □

Similar to [10] we can define the Dennis trace map from $K_*(\mathscr{A})$ to $HH_*(\mathscr{A})$ and the Jones–Goodwillie Chern map from $K_*(\mathscr{A})$ to $HN_*(\mathscr{A})$ by using our $Q$.-construction in place of the $S$.-construction.

The Dennis trace map will be induced by the composite

$$NQ.\mathscr{A} \xrightarrow{id} \bar{C}_0(Q.\mathscr{A}) \subset \bar{C}(Q.\mathscr{A})$$

and the Jones–Goodwillie Chern map will be induced by the composite

$$NQ.\mathscr{A} \xrightarrow{\alpha} Z_0(B\bar{N}(Q.\mathscr{A})) \subset B\bar{N}(Q.\mathscr{A}),$$

where $NQ.\mathscr{A}$ is the nerve of $Q.\mathscr{A}$.

It is clear that by the edgewise subdivision theorem [11, Proposition A.1] and the equivalence between the categories $S_{2n+1}\mathscr{A}$ and $Q_n\mathscr{A}$ for all $n$, the definitions of the Dennis trace map and the Jones–Goodwillie Chern map using the $Q$.-construction, as above, is equivalent to the definitions given by McCarthy using the $S$.-construction. In particular when $A$ is a $k$-algebra and $\mathscr{A}$ is $\mathscr{P}_A$, we get the usual Dennis trace map and Jones–Goodwillie Chern map.

## 2. $U$-theory of exact categories with duality functors

Let $\mathscr{A}$ be an exact category with a duality functor. A *duality functor* is a contravariant exact functor $*:\mathscr{A} \to \mathscr{A}$ such that there is a natural isomorphism $(i_M)_{M\in\mathscr{A}}: id \to * \circ *$ with $i_M^* \circ i_{M^*} = id_{M^*}$ for all $M \in \mathscr{A}$. Fix $\varepsilon = \pm 1$. An $\varepsilon$-hermitian module in $\mathscr{A}$ is a pair $(M, h)$ with $M \in \mathscr{A}$ and $h: M \to M^*$ such that $h = \varepsilon h^*$. $(M, h)$ is called *non-singular* if $h$ is an isomorphism.

Based on Giffen's unpublished idea, Uridia in [14] defines the $U$-theory of an exact category with a duality functor as follows. Let ${}_\varepsilon W\mathscr{A}$ be the category whose objects are all non-singular $\varepsilon$-hermitian modules. A morphism in ${}_\varepsilon W\mathscr{A}$ from $(M, h)$ to $(N, g)$ is an equivalence class of the diagram $M \overset{j}{\twoheadleftarrow} M' \overset{i}{\rightarrowtail} N$ where $i$ (resp. $j$) is an admissible monomorphism (resp. epimorphism) in $\mathscr{A}$ such that the following is a bicartesian square in $\mathscr{A}$:

$$\begin{array}{ccc} M' & \overset{i}{\rightarrowtail} & N \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i^*\circ g} \\ M & \overset{j^*\circ h}{\rightarrowtail} & (M')^* \end{array}$$

Two diagrams $M \twoheadleftarrow M' \rightarrowtail N$ and $M \twoheadleftarrow M'' \rightarrowtail N$ are equivalent if there is an isomorphism $\theta: M' \to M''$ such that the obvious diagram commutes. For any admissible monomorphism $N_1 \rightarrowtail N$ with $(N, g)$ a non-singular $\varepsilon$-hermitian module, let $N_1^\perp$, be the kernel of the composite $N \xrightarrow{g} N^* \twoheadrightarrow N_1^*$. When $N_1 \subset N_1^\perp$, we call $N_1$ a *total isotropic* subobject of $N$.

In the above bicartesian square, if we let $N_1 = \ker j$, then $N_1^\perp = M'$. Hence any morphism in ${}_\varepsilon W\mathscr{A}$ from $(M, h)$ to $(N, g)$ can be identified by $(N_1, \varphi)$, where $N_1 \subset N$ is a total isotropic subobject of $N$ and $\varphi: (N_1^\perp/N_1, g) \to (M, h)$ is an isomorphism of $\varepsilon$-hermitian modules.

**Definition** (Uridia [14]). Let $\mathscr{A}$ be an exact category with a duality functor. The ${}_\varepsilon U$-theory of $\mathscr{A}$ is defined to be the loop space of the classifying space of ${}_\varepsilon W\mathscr{A}$, i.e., ${}_\varepsilon U(\mathscr{A}) = \Omega|{}_\varepsilon W\mathscr{A}|$. Denote ${}_\varepsilon U_n(\mathscr{A}) = \pi_n({}_\varepsilon U(\mathscr{A})) = \pi_{n+1}({}_\varepsilon W\mathscr{A})$.

Charney and Lee [1] show that if $A$ is a $k$-algebra, $\frac{1}{2} \in k$ and $\mathscr{P}_A$ is the exact category of finitely generated projective $A$-modules with the duality functor $* = Hom(-, A)$, then there is a homotopy fibration sequence

$${}_\varepsilon U(\mathscr{P}_A) \to K(A) \to {}_\varepsilon L(A).$$

So ${}_\varepsilon U(\mathscr{P}_A)$ is equivalent to the ${}_\varepsilon U(A)$ defined by Karoubi in [6].

**Remark 1.** ${}_\varepsilon W\mathscr{A}$ is not connected in general. We know that $\Omega|{}_\varepsilon W\mathscr{A}|$ only depends on the subcategory of ${}_\varepsilon W\mathscr{A}$ made up of all the metabolic $\varepsilon$-hermitian modules. The classifying space of this subcategory is the connected component containing 0. One reason, besides others, for studying the full category ${}_\varepsilon W\mathscr{A}$ is that $\pi_0(|{}_\varepsilon W\mathscr{A}|)$ is the usual Witt group.

**Remark 2.** If $\mathscr{A}$ is an exact category with duality functor for which short exact sequences are not necessarily split, then there is currently no definition for the $L$-theory of that category. For these general categories it seems that $U$-theory is more natural than $L$-theory. Also, if needed, we could introduce $L$-theory as the delooping of the homotopy fibre of the map ${}_\varepsilon U(\mathscr{A}) \to K(\mathscr{A})$, which comes from the forgetful functor.

By forgetting the $\varepsilon$-hermitian form, we have a natural functor from ${}_\varepsilon W\mathscr{A}$ to $Q\mathscr{A}$, and hence a map $N_\varepsilon W\mathscr{A} \to NQ\mathscr{A}$. Composing this with the Dennis trace map and the Jones–Goodwillie Chern map we generalize these maps into $U$-theory:

$$D: {}_\varepsilon U_*(\mathscr{A}) \to HH_*(\mathscr{A}),$$

$$J - G: {}_\varepsilon U_*(\mathscr{A}) \to HN_*(\mathscr{A}).$$

In Section 4 we will show that when $\frac{1}{2} \in k$, $D$ and $J$–$G$ factor through involutive homologies. The definitions of these homologies will be given in Section 3.

## 3. Involutive homologies

The involutive homologies considered below are more or less well known, several authors considered them and used slightly different notations (see, e.g., [3, 7–9]). Let $E$ be a chain complex of $k$-modules. An *involution* on $E$ is a chain map $y: E \to E$ such that $y^2 = id_E$. Let ${}_+E^h$ denote the total complex of the double complex:

$$E \xleftarrow{1-y} E \xleftarrow{1+y} E \xleftarrow{1-y} E \xleftarrow{1+y} \cdots$$

and let ${}_{-}E^h$ denote the total complex of the double complex:

$$E \xleftarrow{1+y} E \xleftarrow{1-y} E \xleftarrow{1+y} E \xleftarrow{1-y} \cdots .$$

We call the homology of ${}_{+}E^h$ the *plus-involutive homology* of $E$ and denote it by ${}_{+}H_*^{inv}(E)$, and call the homology of ${}_{-}E^h$ the *minus-involutive homology* of $E$ and denote it by ${}_{-}H_*^{inv}(E)$ (${}_{+}H_*^{inv}(E)$ and ${}_{-}H_*^{inv}(E)$ can be understood in terms of group homology of $Z/2$, see [7]). There is a natural map from $H_*(E)$ to ${}_{+}H_*^{inv}(E)$ (resp. ${}_{-}H_*^{inv}(E)$) induced by the map sending $E$ to the first column of ${}_{+}E^h$ (resp. ${}_{-}E^h$).

When $\frac{1}{2} \in k$, we have maps:

$$E \xrightarrow{(1-y)/2} E \quad \text{and} \quad E \xleftarrow{(1+y)/2} E,$$

with $\frac{1}{2}(1-y) + \frac{1}{2}(1+y) = id_E$. So $E = Im((1-y)/2) \oplus Im((1+y)/2)$ and the two summands are quasi-isomorphic to ${}_{-}E^h$ and ${}_{+}E^h$, respectively. Hence

$$H_*(E) = {}_{+}H_*^{inv}(E) \oplus {}_{-}H_*^{inv}(E).$$

The following is a well known fact.

**Lemma 3.1.** *Let $M$ and $N$ be two double complexes and $\alpha: M \to N$ a chain map. If on each column (row) $\alpha: M_{p,*} \to N_{p,*}(\alpha: M_{*,q} \to N_{*,q})$ is a quasi-isomorphism then $\alpha$ is a quasi-isomorphism.*

**Lemma 3.2.** *Let $E$ and $F$ be two chain complexes and suppose $y$ is an involution on $E$, $z$ is involution on $F$. If $\alpha: E \to F$ is a quasi-isomorphism such that $\alpha \circ y = z \circ \alpha$, then $\alpha: {}_{+}E^h \to {}_{+}F^h$ and $\alpha: {}_{-}E^h \to {}_{-}F^h$ are also quasi-isomorphisms.*

**Proof.** Since $\alpha$ is a quasi-isomorphism from each column of ${}_{+}E^h$ to the corresponding column of ${}_{+}F^h$, by Lemma 3.1, ${}_{+}E^h$ and ${}_{+}F^h$ are quasi-isomorphic. The same is for ${}_{-}E^h$ and ${}_{-}F^h$. □

**Lemma 3.3.** *Let $E$ and $F$ be two chain complexes and $y$ an involution on $F$. If $\alpha: E \to F$ is a chain map such that $\alpha$ and $y \circ \alpha$ are chain homotopic, then the composite $H_*(E) \xrightarrow{\alpha_*} H_*(F) \to {}_{-}H_*^{inv}(F)$ sends $H_*(E)$ into the 2-primary part of ${}_{-}H_*^{inv}(F)$. In particular if $\frac{1}{2} \in k$, then the composite map is zero, which means that $Im(\alpha_*) \subset {}_{+}H_*^{inv}(F)$.*

**Proof.** First assume $\frac{1}{2} \in k$. Since $\alpha$ and $y \circ \alpha$ are chain homotopic we have $\alpha_* = y_* \circ \alpha_*: H_*(E) \to H_*(F)$, e.g. $(1-y)_* \circ \alpha_* = 0$. We have

$$H_*(F) = Im(1+y)_* \oplus Im(1-y)_* = {}_{+}H_*^{inv}(F) \oplus {}_{-}H_*^{inv}(F).$$

Therefore we have $\alpha_*(H_*(E)) \subset {}_{+}H_*^{inv}(F)$.

For the case where $\frac{1}{2}$ is not an element of $k$, simply localize at 2. By the above calculation, the composite $H_*(E) \to H_*(F) \to {}_-H_*^{inv}(F)$, after localizing, is the zero map. In particular, the composite sends $H_*(E)$ to the 2-primary part of ${}_-H_*^{inv}(F)$. □

Let $A$ be a $k$-algebra with an involution $a \to \bar{a}$. Define $y : \bar{C}_n(A) \to \bar{C}_n(A)$ by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \to (-1)^{n(n+1)/2} \bar{a}_0 \otimes \bar{a}_n \otimes \cdots \otimes \bar{a}_1.$$

Loday [7] shows that $y \circ b = b \circ y$ and $y \circ B = -B \circ y$. This means that $y : \bar{C}_n(A) \to \bar{C}_n(A)$ is an involution. We call ${}_+HH_*^{inv}(A) = {}_+H_*^{inv}(\bar{C}(A))$ the *plus involutive Hochschild homology* (in terms of the group homology of $Z/2$, it is called the $Z/2$-equivariant Hochschild homology, see [7]).

In order that $y$ induces an involution on $B\bar{P}(A)$, $B\bar{C}(A)$, $B\bar{N}(A)$ we need to adjust the sign of $y$. The new action, which we denote as $z$, is $(-1)^p y$ on the $p$th column (see [3]). ${}_+H_*^{inv}(B\bar{C}(A))$ is called the *dihedral homology* of $A$. We will call ${}_+H_*^{inv}(B\bar{P}(A))$ and ${}_+H_*^{inv}(B\bar{N}(A))$ the *plus involutive periodic* and *negative homology* of $A$ (the last is also called quadrant II dihedral homology, see [7]).

The construction of the above involutive homologies can be extended to define involutive homologies for an exact $k$-linear category with a duality functor. Suppose $\mathscr{A}$ is a $k$-linear category with a duality functor. In Section 1 we described the chain complexes $\bar{C}(\mathscr{A})$, $B\bar{P}(\mathscr{A})$, $B\bar{C}(\mathscr{A})$, $B\bar{N}(\mathscr{A})$. As in the ring case we can define $y : \bar{C}(\mathscr{A}) \to \bar{C}(\mathscr{A})$ by

$$f_0 \otimes f_1 \otimes \cdots \otimes f_n \to (-1)^{n(n+1)/2} f_0^* \otimes f_n^* \otimes \cdots \otimes f_1^*.$$

it is easy to check that $y \circ b = b \circ y$ and $B \circ y = -y \circ B$, which means that $y$ is an involution on $\bar{C}(\mathscr{A})$. Once again by the same sign provision, we can induce the involution $z$ on $B\bar{P}(\mathscr{A})$, $B\bar{C}(\mathscr{A})$ and $B\bar{N}(\mathscr{A})$.

Now let $\mathscr{A}$ be an exact $k$-linear category with a duality functor. Then the duality functor of $\mathscr{A}$ induces a duality functor on $Q_n\mathscr{A}$, for all $n$. More precisely, for a morphism $f : M \to N$ in $Q\mathscr{A}$, if $f$ is represented by $M \twoheadleftarrow M' \rightarrowtail N$, then let the map $f^* : M^* \to N^*$ in $Q\mathscr{A}$ denote the map represented by $M^* \rightarrowtail M'^* \twoheadleftarrow N^*$. Note that this does not induce a duality functor on $Q\mathscr{A}$, since the above is not from $N^*$ to $M^*$. But we can use it to induce a duality functor on $Q_n\mathscr{A}$. For an object $M$ in $Q_n\mathscr{A}$,

$$M =: M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} M_n$$

define the dual of that object as

$$M^* =: M_0^* \xrightarrow{f_1^*} M_1^* \xrightarrow{f_2^*} \cdots \xrightarrow{f_n^*} M_n^*$$

For a morphism $(a_0, \ldots, a_n) : M \to P$, we define the dual as $(a_0^*, \ldots, a_n^*) : P^* \to M^*$. Clearly this defines a duality functor on $Q_n\mathscr{A}$.

Now that $Q_n\mathscr{A}$ is a $k$-linear category with a duality functor, we have the involutions $y$ and $z$ on $\bar{C}(Q_n\mathscr{A})$, $B\bar{C}(Q_n\mathscr{A})$, $B\bar{P}(Q_n\mathscr{A})$ and $B\bar{N}(Q_n\mathscr{A})$. This duality functor is also compatible with the simplicial structure of $Q.\mathscr{A}$, so the induced involutions are also compatible with the simplicial structures of $\bar{C}(Q.\mathscr{A})$, $B\bar{P}(Q.\mathscr{A})$, $B\bar{C}(Q.\mathscr{A})$ and $B\bar{N}(Q.\mathscr{A})$.

**Definition.** If $\mathscr{A}$ is an exact $k$-linear category with a duality functor, then we define:

$${}_{+}HH_{*}^{inv}(\mathscr{A}) = {}_{+}H_{*+1}^{inv}(\bar{C}(Q.\mathscr{A}))\ \ + \text{involutive Hochschild homology of } \mathscr{A},$$

$${}_{+}HP_{*}^{inv}(\mathscr{A}) = {}_{+}H_{*+1}^{inv}(B\bar{P}(Q.\mathscr{A}))\ \ + \text{involutive periodic homology of } \mathscr{A},$$

$${}_{+}HN_{*}^{inv}(\mathscr{A}) = {}_{+}H_{*+1}^{inv}(B\bar{N}(Q.\mathscr{A}))\ \ + \text{involutive negative homology of } \mathscr{A},$$

$${}_{+}HC_{*}^{inv}(\mathscr{A}) = {}_{+}H_{*+1}^{inv}(B\bar{C}(Q.\mathscr{A}))\ \text{dihedral homology of } \mathscr{A}.$$

We will also write *dihedral homology* in the standard notation $HD_{*}(\mathscr{A}) = {}_{+}HC_{*}^{inv}(\mathscr{A})$.

**Remark.** We can also use the $S.$-construction to define the involutive homologies for an exact $k$-linear category $\mathscr{A}$ with a duality functor $*$. The duality functor induces a duality functor for each of the $S_n\mathscr{A}$ by defining $(* \rightarrowtail A_1 \rightarrowtail A_2 \rightarrow \cdots \rightarrowtail A_n)^*$ as

$$* \rightarrowtail (A_n/A_{n-1})^* \rightarrowtail (A_n/A_{n-2})^* \rightarrowtail \cdots \rightarrowtail (A_n/A_1)^* \rightarrowtail A_n^*.$$

Then apply to $S.\mathscr{A}$ the same process used to define the involutive homologies of $Q.\mathscr{A}$. By passing to edgewise subdivision, as in the proof of Proposition 1.1, we see that the involutive homologies of $S.\mathscr{A}$ and $Q.\mathscr{A}$ are the same.

**Proposition 3.1.** *Let $A$ be a $k$-algebra with involution " $-$ ", and let $\mathscr{P}_A$ be the category of all finitely generated projective $A$-modules with duality functor $* = Hom_A(\ , A)$. Then the involutive homologies of $A$ are the same as the involutive homologies of $\mathscr{P}_A$.*

**Proof.** By the above note we can define the involutive homologies of $\mathscr{P}_A$ using $\bar{C}(S.\mathscr{P}_A)$. From [10] we have a chain map from $\bar{C}(A)$ to $\bar{C}(S.\mathscr{P}_A)$ which is a quasi-isomorphism. The map comes from the inclusion

$$\bar{C}_n(A) = A \otimes \bar{A}^{\otimes n} = Hom_A(A, A) \otimes \overline{Hom_A(A, A)}^{\otimes n} \subset \bar{C}_n(S_1\mathscr{P}_A).$$

Furthermore, we can see that the chain map commutes with the involutions on $\bar{C}(A)$ and $\bar{C}(S.\mathscr{P}_A)$. So by Lemma 3.2 we have that the involutive homologies of $A$ are isomorphic to the involutive homologies of $\mathscr{P}_A$. $\square$

## 4. Chern maps

Let $\mathscr{A}$ be an exact $k$-linear category with a duality functor $*$. Then $*$ induces a covariant functor $v: Q\mathscr{A} \rightarrow Q\mathscr{A}$ by sending the object $M$ to $M^*$, and sending the morphism $M \twoheadleftarrow M' \rightarrowtail N$ to $M^* \rightarrowtail M'^* \twoheadleftarrow N^*$. Clearly $v^2 = id$. This induces a self map on the classifying space of $Q\mathscr{A}$ and thus an endomorphism on $K(\mathscr{A})$. We will still denote it as $v$ and call it an *involution on* $K(\mathscr{A})$ (cf. [9]).

Recall that the Dennis trace map and the Jones–Goodwillie Chern map are induced by

$$D: NQ.\mathscr{A} \to \bar{C}_0(Q.\mathscr{A}) \subset \bar{C}(Q.\mathscr{A})$$

and

$$J - G: NQ.\mathscr{A} \to Z_0(B\bar{N}(Q.\mathscr{A})) \subset B\bar{N}(Q.\mathscr{A}).$$

We can see that $y \circ D = D \circ v$ and $z \circ (J - G) = (J - G) \circ v$, i.e. the Dennis trace map and the Jones–Goodwillie Chern map commute with the involutions on $K(\mathscr{A})$, $HH(\mathscr{A})$ and $HN(\mathscr{A})$.

Let $F: {}_\varepsilon W\mathscr{A} \to Q\mathscr{A}$ denote the forgetful functor. We will still use $F$ to denote the map from ${}_\varepsilon U(\mathscr{A}) = \Omega|{}_\varepsilon W\mathscr{A}| \to \Omega|Q\mathscr{A}| = K(\mathscr{A})$ induced by the forgetful functor.

**Lemma 4.1.** *There is a natural isomorphism $i$ between the functor $v \circ F$ and $F: {}_\varepsilon W\mathscr{A} \to Q\mathscr{A}$. So $v \circ F = F: {}_\varepsilon U_*(\mathscr{A}) \to K_*(\mathscr{A})$.*

**Proof.** For any object $(M, h) \in {}_\varepsilon W\mathscr{A}$, we will define the natural isomorphism by $i_{(M,h)} = h: F((M, h)) = M \to M^* = v \circ F((M, h))$. We would like that for any morphism from $(M, h)$ to $(N, g)$ in ${}_\varepsilon W\mathscr{A}$, represented by $M \twoheadleftarrow M' \rightarrowtail N$, the following diagram commutes in $Q\mathscr{A}$:

$$\begin{array}{ccccc} M & \twoheadleftarrow & M' & \rightarrowtail & N \\ {\scriptstyle h}\downarrow & & & & \downarrow{\scriptstyle g} \\ M^* & \rightarrowtail & M'^* & \twoheadleftarrow & N^* \end{array}$$

But this is equal to saying that the following diagram is a bicartesian square:

$$\begin{array}{ccccc} M' & \rightarrowtail & & \longrightarrow & N \\ \Big| & & & & \downarrow{\scriptstyle g} \\ \Big| & & & & N^* \\ \downarrow & & & & \downarrow \\ M & \xrightarrow{h} & M^* & \rightarrowtail & M'^* \end{array}$$

That this is a bicartesian square follows directly from the requirement imposed on morphisms in ${}_\varepsilon W\mathscr{A}$.

The above gives us that $v \circ F$ and $F$ are naturally isomorphic functors, hence the maps they induce on the classifying spaces are homotopic. Therefore $v \circ F = F: {}_\varepsilon U_*(\mathscr{A})) \to K_*(\mathscr{A})$. □

**Theorem 4.2.** *Let $\mathscr{A}$ be an exact $k$-linear category. Then the image of the composite map ${}_\varepsilon U_*(\mathscr{A}) \xrightarrow{D} HH_*(\mathscr{A}) \to {}_-HH_*^{inv}(\mathscr{A})$ lies in the 2-primary part of ${}_-HH_*^{inv}(\mathscr{A})$. In particular if $\frac{1}{2} \in k$, then the Dennis trace map $D$ maps ${}_\varepsilon U_*(\mathscr{A})$ into ${}_+HH_*^{inv}(\mathscr{A}) \subset HH_*(\mathscr{A})$.*

**Proof.** We pointed out at the begining of this section that for the Dennis trace map $D: NQ.\mathscr{A} \to \bar{C}(Q.\mathscr{A})$, we have $y \circ D = D \circ v$, where $y$ is the involution on $\bar{C}(Q.\mathscr{A})$ induced from the involution on $\mathscr{A}$. Putting this together with Lemma 4.1 we have that $D \circ F \simeq y \circ D \circ F: {}_\varepsilon U(\mathscr{A}) \to \bar{C}(Q.\mathscr{A})$. Now using Lemma 3.3, we have the results of our theorem. □

Using the same proof, with $J-G$ in place of $D$, we get the following:

**Theorem 4.3.** *Let $\mathscr{A}$ be an exact k-linear category. The image of the composite ${}_\varepsilon U_*(\mathscr{A}) \xrightarrow{J-G} HN_*(\mathscr{A}) \to {}_-HN_*^{inv}(\mathscr{A})$ lies in the 2-primary part of ${}_-HN_*^{inv}(\mathscr{A})$. In particular if $\frac{1}{2} \in k$, then the Jones–Goodwillie Chern map J–G maps ${}_\varepsilon U_*(\mathscr{A})$ into ${}_+HN_*^{inv}(\mathscr{A}) \subset HN_*(\mathscr{A})$.*

Let $x_p$ be the composite of chain maps

$$B\bar{N}(Q.\mathscr{A}) \subset B\bar{P}(Q.\mathscr{A}) = \varprojlim_n B\bar{C}(Q.\mathscr{A})[2n] \to B\bar{C}(Q.\mathscr{A})[2p]$$

Clearly $x_p$ commutes with the involutions. This gives us the induced maps:

$$x_p: HN_*(\mathscr{A}) \to HC_{*+2p}(\mathscr{A}),$$

$$_+x_p: {}_+HN_*^{inv}(\mathscr{A}) \to {}_+HC_{*+2p}^{inv}(\mathscr{A}) = HD_{*+2p}(\mathscr{A}),$$

$$_-x_p: {}_-HN_*^{inv}(\mathscr{A}) \to {}_-HC_{*+2}^{inv}(\mathscr{A}).$$

When $\frac{1}{2} \in k$, we have $x_p = {}_+x_p \oplus {}_-x_p$.

Consider the composite: ${}_\varepsilon U_*(\mathscr{A}) \xrightarrow{J-G} HN_*(\mathscr{A}) \xrightarrow{x_p} HC_{*+2p}(\mathscr{A})$. We will still call it a Jones–Goodwillie Chern map. From Theorem 4.3 we have:

**Corollary 4.4.** *Let $\mathscr{A}$ be an exact k-linear category. Then the composite ${}_\varepsilon U_*(\mathscr{A}) \xrightarrow{J-G} HC_{*+2p}(\mathscr{A}) \to {}_{-1}HC_{*+2p}^{inv}(\mathscr{A})$ has its image in the 2-primary part of ${}_{-1}HC_{*+2p}^{inv}(\mathscr{A})$. When $\frac{1}{2} \in k$, the Jones–Goodwillie Chern maps send ${}_\varepsilon U_*(\mathscr{A})$ into $HD_{*+2p}(\mathscr{A}) \subset HC_{*+2p}(\mathscr{A})$.*

Suppose $\mathscr{A}$ is an exact category with a duality functor in which all short exact sequences are split. As Karoubi did in the ring case we can mimick Quillen's $S^{-1}S$-construction to define the $L$-theory for $\mathscr{A}$. There is also an analogy of the $Q$-construction which can be used to define the $L$-theory for such a category, and the two definitions by $Q$ and $S^{-1}S$ constructions are equivalent (see [13]). In more detail, let $\mathscr{A}$ be an exact category with a duality functor and let $\varepsilon = \pm 1$. Define ${}_\varepsilon Q\mathscr{A}$ in the following way. The objects are non-singular $\varepsilon$-hermitian modules $(M, h)$ and a morphism $(M,h) \to (N,g)$ is given by $((M_1,h_1), (M_2,h_2), \varphi)$, where $(M_1,h_1)$, $(M_2,h_2) \in {}_\varepsilon Q\mathscr{A}$ and $\varphi: (M,h) \oplus (M_1,h_1) \oplus (M_2,h_2) \to (N,g)$ is an isomorphism of $\varepsilon$-hermitian modules. Then the $L$-theory of $\mathscr{A}$ is defined by ${}_\varepsilon L(\mathscr{A}) = \Omega |{}_\varepsilon Q\mathscr{A}|$.

The forgetful functor $F$ from ${}_\varepsilon Q\mathscr{A}$ to $Q\mathscr{A}$, which simply forgets the $\varepsilon$-hermitian forms, induces a map ${}_\varepsilon L(\mathscr{A}) \to K(\mathscr{A})$. Using a similar proof to the one used for $U$-theory, we have:

**Theorem 4.5.** *Let $\mathscr{A}$ be an exact k-linear category with a duality functor such that all short exact sequences split and $\frac{1}{2} \in k$. Then the image of the composite ${}_\varepsilon L_*(\mathscr{A}) \xrightarrow{F} K_*(\mathscr{A}) \xrightarrow{D} HH_*(\mathscr{A}) = {}_+HH_*^{inv}(\mathscr{A}) \oplus {}_-HH_*^{inv}(\mathscr{A})$ lies in ${}_+HH_*^{inv}(\mathscr{A})$ and the image of the composite ${}_\varepsilon L_*(\mathscr{A}) \xrightarrow{F} K_*(\mathscr{A}) \xrightarrow{J-G} HC_{*+2p}(\mathscr{A}) = {}_+HC_{*+2p}^{inv}(\mathscr{A}) \oplus {}_-HC_{*+2p}^{inv}(\mathscr{A})$ lies in ${}_+HC_{*+2p}^{inv}(\mathscr{A}) = HD_{*+2p}(\mathscr{A})$. Similar statements can be made for $HN_*$ and $HP_*$.*

**Proof.** The key here is to prove an analogy to Lemma 4.1. That is, we must show that the functors $v \circ F$ and $F$ from ${}_\varepsilon Q\mathscr{A}$ to $Q\mathscr{A}$ are naturally isomorphic. For any $(M, h) \in {}_\varepsilon Q\mathscr{A}$, let the natural isomorphism $i_{(M,h)} = h : F((M, h)) = M \to M^* = v \circ F((M, h))$. To see naturality let us note that for any morphism from $(M, h)$ to $(N, g)$, given by $((M_1, h_1), (M_2, h_2), \varphi)$, the following diagram in $Q\mathscr{A}$ commutes:

$$\begin{array}{ccccccc} M & \twoheadleftarrow & M \oplus M_1 & \rightarrowtail & M \oplus M_1 \oplus M_2 & \xrightarrow[\cong]{\varphi} & N \\ {\scriptstyle h}\downarrow{\scriptstyle \cong} & & & & & & {\scriptstyle g}\downarrow{\scriptstyle \cong} \\ M^* & \rightarrowtail & M^* \oplus M_1^* & \twoheadleftarrow & M^* \oplus M_1^* \oplus M_2^* & \xleftarrow[\cong]{\varphi^*} & N^* \end{array}$$

**Remark.** As we mentioned in the introduction, this provides another proof of a result in [2, 9].

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